

# Limits of Preprocessing\*

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## Abstract

We present a first theoretical analysis of the power of polynomial-time preprocessing for important combinatorial problems from various areas in AI. We consider problems from Constraint Satisfaction, Global Constraints, Satisfiability, Nonmonotonic and Bayesian Reasoning. We show that, subject to a complexity theoretic assumption, none of the considered problems can be reduced by polynomial-time preprocessing to a problem kernel whose size is polynomial in a structural problem parameter of the input, such as induced width or backdoor size. Our results provide a firm theoretical boundary for the performance of polynomial-time preprocessing algorithms for the considered problems.

**Keywords:** Fixed-Parameter Tractability, Constraint Satisfaction, Global Constraints, Satisfiability, Bayesian Networks, Normal Logic Programs, Computational Complexity.

## 1 Introduction

Many important computational problems that arise in various areas of AI are intractable. Nevertheless, AI research was very successful in developing and implementing heuristic solvers that work well on real-world instances. An important component of virtually every solver is a powerful polynomial-time preprocessing procedure that reduces the problem input. For instance, preprocessing techniques for the propositional satisfiability problem are based on Boolean Constraint Propagation (see, e.g., Eén and Biere, 2005), CSP solvers make use of various local consistency algorithms that filter the domains of variables (see, e.g., Bessière, 2006); similar preprocessing methods are used by solvers for Nonmonotonic and Bayesian reasoning problems (see, e.g., Gebser et al., 2008, Bolt and van der Gaag, 2006, respectively).

Until recently, no provable performance guarantees for polynomial-time preprocessing methods have been obtained, and so preprocessing was only subject of empirical studies. A possible reason for the lack of theoretical results is a certain inadequacy of the P vs NP framework for such an analysis: if we could reduce in polynomial time an instance of an NP-hard problem just by one bit, then we can solve the entire problem in polynomial time by repeating the reduction step a polynomial number of times, and  $P = NP$  follows.

With the advent of *parameterized complexity* (Downey, Fellows, and Stege, 1999), a new theoretical framework became available that provides suitable tools to analyze the power of preprocessing. Parameterized complexity considers a problem in a two-dimensional setting, where in addition to the input size  $n$ , a *problem parameter*  $k$  is taken into consideration. This parameter can encode a structural aspect of the problem instance. A problem is called *fixed-parameter tractable* (FPT) if it can be solved in time  $f(k)p(n)$  where  $f$  is a function of the parameter  $k$  and  $p$  is a polynomial of the input size  $n$ . Thus, for FPT problems, the combinatorial explosion can be confined to the parameter and is independent of the input size. It is known that a problem is fixed-parameter tractable if and only if every problem input can be reduced by polynomial-time preprocessing to an equivalent input whose size is bounded by a function of the parameter (Downey, Fellows, and Stege, 1999). The reduced instance is called the *problem kernel*, the preprocessing is called *kernelization*. The power of polynomial-time preprocessing can now be benchmarked in terms of the size of the kernel. Once a small kernel is obtained, we can apply any method

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\*Research funded by the ERC (COMPLEX REASON, Grand Reference 239962).

of choice to solve the kernel: brute-force search, heuristics, approximation, etc. (Guo and Niedermeier, 2007). Because of this flexibility a small kernel is generally preferable to a less flexible branching-based fixed-parameter algorithm. Thus, small kernels provide an additional value that goes beyond bare fixed-parameter tractability.

In general the size of the kernel is exponential in the parameter, but many important NP-hard optimization problems such as Minimum Vertex Cover, parameterized by solution size, admit *polynomial kernels*, see, e.g., (Bodlaender et al., 2009) for references.

In previous research several NP-hard AI problems have been shown to be fixed-parameter tractable. We list some important examples from various areas:

- Constraint satisfaction problems (CSP) over a fixed universe of values, parameterized by the induced width (Gottlob, Scarcello, and Sideri, 2002).
- Consistency and generalized arc consistency for intractable global constraints, parameterized by the cardinalities of certain sets of values (Bessière et al., 2008).
- Propositional satisfiability (SAT), parameterized by the size of backdoors (Nishimura, Ragde, and Szeider, 2004).
- Positive inference in Bayesian networks with variables of bounded domain size, parameterized by size of loop cutsets (Pearl, 1988; Bidyuk and Dechter, 2007).
- Nonmonotonic reasoning with normal logic programs, parameterized by feedback width (Gottlob, Scarcello, and Sideri, 2002).

However, only exponential kernels are known for these fundamental AI problems. Can we hope for polynomial kernels?

**Results** Our results are throughout negative. We provide strong theoretical evidence that none of the above fixed-parameter tractable AI problems admits a polynomial kernel. More specifically, we show that a polynomial kernel for any of these problems causes a collapse of the Polynomial Hierarchy to its third level, which is considered highly unlikely by complexity theorists.

Our results are general: The kernel lower bounds are not limited to a particular preprocessing technique but apply to any clever technique that could be conceived in future research. Hence the results contribute to the foundations of AI.

Our results suggest the investigation of alternative approaches to polynomial-time preprocessing; for instance, preprocessing that produces in polynomial time a Boolean combination of polynomially sized kernels instead of one single kernel.

## 2 Formal Background

A *parameterized problem*  $\mathbf{P}$  is a subset of  $\Sigma^* \times \mathbb{N}$  for some finite alphabet  $\Sigma$ . For a problem instance  $(x, k) \in \Sigma^* \times \mathbb{N}$  we call  $x$  the main part and  $k$  the parameter. We assume the parameter is represented in unary. For the parameterized problems considered in this paper, the parameter is a function of the main part, i.e.,  $k = \pi(x)$  for a function  $\pi$ . We then denote the problem as  $\mathbf{P}(\pi)$ , e.g.,  $U\text{-CSP}(\text{width})$  denotes the problem  $U\text{-CSP}$  parameterized by the width of the given tree decomposition.

A parameterized problem  $\mathbf{P}$  is *fixed-parameter tractable* if there exists an algorithm that solves any input  $(x, k) \in \Sigma^* \times \mathbb{N}$  in time  $O(f(k) \cdot p(|x|))$  where  $f$  is an arbitrary computable function of  $k$  and  $p$  is a polynomial in  $n$ .

A *kernelization* for a parameterized problem  $\mathbf{P} \subseteq \Sigma^* \times \mathbb{N}$  is an algorithm that, given  $(x, k) \in \Sigma^* \times \mathbb{N}$ , outputs in time polynomial in  $|x| + k$  a pair  $(x', k') \in \Sigma^* \times \mathbb{N}$  such that (i)  $(x, k) \in \mathbf{P}$  if and only if  $(x', k') \in \mathbf{P}$  and (ii)  $|x'| + k' \leq g(k)$ , where  $g$  is an arbitrary computable function, called the *size* of the kernel. In particular, for constant  $k$  the kernel has constant size  $g(k)$ . If  $g$  is a polynomial then we say that  $\mathbf{P}$  admits a *polynomial kernel*.

Every fixed-parameter tractable problem admits a kernel. This can be seen by the following argument due to Downey, Fellows, and Stege (1999). Assume we can decide instances  $(x, k)$  of problem  $\mathbf{P}$  in time  $f(k)|n|^{O(1)}$ . We kernelize an instance  $(x, k)$  as follows. If  $|x| \leq f(k)$  then we already have a kernel of size  $f(k)$ . Otherwise, if  $|x| > f(k)$ , then  $f(k)|x|^{O(1)} \leq |x|^{O(1)}$  is a polynomial; hence we can decide the instance in polynomial time and replace it with a small decision-equivalent instance  $(x', k')$ . Thus we always have a kernel of size at most  $f(k)$ . However,  $f(k)$  is super-polynomial for NP-hard problems (unless  $\mathbf{P} = \mathbf{NP}$ ), hence this generic construction is not providing polynomial kernels.

We understand *preprocessing* for an NP-hard problem as a *polynomial-time* procedure that transforms an instance of the problem to a (possibly smaller) solution-equivalent instance of the same problem. Kernelization is such a preprocessing with a *performance guarantee*, i.e., we are guaranteed that the preprocessing yields a kernel whose size is bounded in terms of the parameter of the given problem instance. In the literature also different forms of preprocessing have been considered. An important one is *knowledge compilation*, a two-phases approach to reasoning problems where in a first phase a given knowledge base is (possibly in exponential time) preprocessed (“compiled”), such that in a second phase various queries can be answered in polynomial time (Cadoli et al., 2002).

### 3 Tools for Kernel Lower Bounds

In the sequel we will use recently developed tools to obtain kernel lower bounds. Our kernel lower bounds are subject to the widely believed complexity theoretic assumption  $\mathbf{NP} \not\subseteq \text{co-NP/poly}$  (or equivalently,  $\mathbf{PH} \neq \Sigma_p^3$ ). In other words, the tools allow us to show that a parameterized problem does not admit a polynomial kernel unless the Polynomial Hierarchy collapses to its third level (see, e.g., Papadimitriou, 1994).

A *composition algorithm* for a parameterized problem  $\mathbf{P} \subseteq \Sigma^* \times \mathbb{N}$  is an algorithm that receives as input a sequence  $(x_1, k), \dots, (x_t, k) \in \Sigma^* \times \mathbb{N}$ , uses time polynomial in  $\sum_{i=1}^t |x_i| + k$ , and outputs  $(y, k') \in \Sigma^* \times \mathbb{N}$  with (i)  $(y, k') \in \mathbf{P}$  if and only if  $(x_i, k) \in \mathbf{P}$  for some  $1 \leq i \leq t$ , and (ii)  $k'$  is polynomial in  $k$ . A parameterized problem is *compositional* if it has a composition algorithm. With each parameterized problem  $\mathbf{P} \subseteq \Sigma^* \times \mathbb{N}$  we associate a classical problem

$$\text{UP}[\mathbf{P}] = \{x\#1^k : (x, k) \in \mathbf{P}\}$$

where  $1$  denotes an arbitrary symbol from  $\Sigma$  and  $\#$  is a new symbol not in  $\Sigma$ . We call  $\text{UP}[\mathbf{P}]$  the *unparameterized version* of  $\mathbf{P}$ .

The following result is the basis for our kernel lower bounds.

**Theorem 1** (Bodlaender et al., 2009, Fortnow and Santhanam, 2008). *Let  $\mathbf{P}$  be a parameterized problem whose unparameterized version is NP-complete. If  $\mathbf{P}$  is compositional, then it does not admit a polynomial kernel unless  $\mathbf{NP} \subseteq \text{co-NP/poly}$ , i.e., the Polynomial Hierarchy collapses.*

Let  $\mathbf{P}, \mathbf{Q} \subseteq \Sigma^* \times \mathbb{N}$  be parameterized problems. We say that  $\mathbf{P}$  is *polynomial parameter reducible* to  $\mathbf{Q}$  if there exists a polynomial time computable function  $K : \Sigma^* \times \mathbb{N} \rightarrow \Sigma^* \times \mathbb{N}$  and a polynomial  $p$ , such that for all  $(x, k) \in \Sigma^* \times \mathbb{N}$  we have (i)  $(x, k) \in \mathbf{P}$  if and only if  $K(x, k) = (x', k') \in \mathbf{Q}$ , and (ii)  $k' \leq p(k)$ . The function  $K$  is called a *polynomial parameter transformation*.

The following theorem allows us to transform kernel lower bounds from one problem to another.

**Theorem 2** (Bodlaender, Thomassé, and Yeo, 2009). *Let  $\mathbf{P}$  and  $\mathbf{Q}$  be parameterized problems such that  $\text{UP}[\mathbf{P}]$  is NP-complete,  $\text{UP}[\mathbf{Q}]$  is in NP, and there is a polynomial parameter transformation from  $\mathbf{P}$  to  $\mathbf{Q}$ . If  $\mathbf{Q}$  has a polynomial kernel, then  $\mathbf{P}$  has a polynomial kernel.*

### 4 Constraint Networks

*Constraint networks* have proven successful in modeling everyday cognitive tasks such as vision, language comprehension, default reasoning, and abduction, as well as in applications such as scheduling, design, diagnosis, and temporal and spatial reasoning (Dechter, 2010). A *constraint network* is a triple  $I =$

$(V, U, \mathcal{C})$  where  $V$  is a finite set of variables,  $U$  is a finite universe of values, and  $\mathcal{C} = \{C_1, \dots, C_m\}$  is set of constraints. Each constraint  $C_i$  is a pair  $(S_i, R_i)$  where  $S_i$  is a list of variables of length  $r_i$  called the *constraint scope*, and  $R_i$  is an  $r_i$ -ary relation over  $U$ , called the *constraint relation*. The tuples of  $R_i$  indicate the allowed combinations of simultaneous values for the variables  $S_i$ . A *solution* is a mapping  $\tau : V \rightarrow U$  such that for each  $1 \leq i \leq m$  and  $S_i = (x_1, \dots, x_{r_i})$ , we have  $(\tau(x_1), \dots, \tau(x_{r_i})) \in R_i$ . A constraint network is *satisfiable* if it has a solution.

With a constraint network  $I = (V, U, \mathcal{C})$  we associate its *constraint graph*  $G = (V, E)$  where  $E$  contains an edge between two variables if and only if they occur together in the scope of a constraint. A *width  $w$  tree decomposition* of a graph  $G$  is a pair  $(T, \lambda)$  where  $T$  is a tree and  $\lambda$  is a labeling of the nodes of  $T$  with sets of vertices of  $G$  such that the following properties are satisfied: (i) every vertex of  $G$  belongs to  $\lambda(p)$  for some node  $p$  of  $T$ ; (ii) every edge of  $G$  is contained in  $\lambda(p)$  for some node  $p$  of  $T$ ; (iii) For each vertex  $v$  of  $G$  the set of all tree nodes  $p$  with  $v \in \lambda(p)$  induces a connected subtree of  $T$ ; (iv)  $|\lambda(p)| - 1 \leq w$  holds for all tree nodes  $p$ . The *treewidth* of  $G$  is the smallest  $w$  such that  $G$  has a width  $w$  tree decomposition. The *induced width* of a constraint network is the treewidth of its constraint graph (Dechter and Pearl, 1989). We note in passing that the problem of finding a tree decomposition of width  $w$  is NP-hard but fixed-parameter tractable in  $w$ .

Let  $U$  be a fixed universe containing at least two elements. We consider the following parameterized version of the constraint satisfaction problem (CSP).

**$U$ -CSP(width)**

*Instance:* A constraint network  $I = (V, U, \mathcal{C})$  and a width  $w$  tree decomposition of the constraint graph of  $I$ .

*Parameter:* The integer  $w$ .

*Question:* Is  $I$  satisfiable?

It is well known that  $U$ -CSP(width) is fixed-parameter tractable over any fixed universe  $U$  (Dechter and Pearl, 1989; Gottlob, Scarcello, and Sideri, 2002) (for generalizations see Samer and Szeider, 2010). We contrast this classical result and show that it is unlikely that  $U$ -CSP(width) admits a polynomial kernel, even in the simplest case where  $U = \{0, 1\}$ .

**Theorem 3.**  $\{0, 1\}$ -CSP(width) does not admit a polynomial kernel unless the Polynomial Hierarchy collapses.

*Proof.* We show that  $\{0, 1\}$ -CSP(width) is compositional. Let  $(I_i, T_i)$ ,  $1 \leq i \leq t$ , be a given sequence of instances of  $\{0, 1\}$ -CSP(width) where  $I_i = (V_i, U_i, \mathcal{C}_i)$  is a constraint network and  $T_i$  is a width  $w$  tree decomposition of the constraint graph of  $I_i$ . We may assume, w.l.o.g., that  $V_i \cap V_j = \emptyset$  for  $1 \leq i < j \leq t$  (otherwise we can simply change the names of variables). We form a new constraint network  $I = (V, \{0, 1\}, \mathcal{C})$  as follows. We put  $V = \bigcup_{i=1}^t V_i \cup \{a_1, \dots, a_t, b_0, \dots, b_t\}$  where  $a_i, b_i$  are new variables. We define the set  $\mathcal{C}$  of constraints in three groups.

(1) For each  $1 \leq i \leq t$  and each constraint  $C = ((x_1, \dots, x_r), R) \in \mathcal{C}_i$  we add to  $\mathcal{C}$  a new constraint  $C' = ((x_1, \dots, x_r, a_i), R')$  where  $R' = \{(u_1, \dots, u_r, 0) : (u_1, \dots, u_r) \in R\} \cup \{(1, \dots, 1)\}$ .

(2) We add  $t$  ternary constraints  $C_1^*, \dots, C_t^*$  where  $C_i^* = ((b_{i-1}, b_i, a_i), R^*)$  and  $R^* = \{(0, 0, 1), (0, 1, 0), (1, 1, 1)\}$ .

(3) Finally, we add two unary constraints  $C^0 = ((b_0), (0))$  and  $C^1 = ((b_t), (1))$  which force the values of  $b_0$  and  $b_t$  to 0 and 1, respectively.

Let  $G, G_i$  be the constraint graphs of  $I$  and  $I_i$ , respectively. Fig. 1 shows an illustration of  $G$  for  $t = 4$ . We observe that  $a_1, \dots, a_t$  are cut vertices of  $G$ . Removing these vertices separates  $G$  into independent parts  $P, G'_1, \dots, G'_t$  where  $P$  is the path  $b_0, b_1, \dots, b_t$ , and  $G'_i$  is isomorphic to  $G_i$ . By standard techniques (see, e.g., Kloks, 1994), we can put the given width  $w$  tree decompositions  $T_1, \dots, T_t$  of  $G'_1, \dots, G'_t$  and the trivial width 1 tree decomposition of  $P$  together to a width  $w + 1$  tree decomposition  $T$  of  $G$ . Clearly  $(I, T)$  can be obtained from  $(I_i, T_i)$ ,  $1 \leq i \leq t$ , in polynomial time.

We claim that  $I$  is satisfiable if and only if at least one of the  $I_i$  is satisfiable. This claim can be verified by means of the following observations: The constraints in groups (2) and (3) provide that for any satisfying assignment there will be some  $0 \leq i \leq t - 1$  such that  $b_0, \dots, b_i$  are all set to 0 and

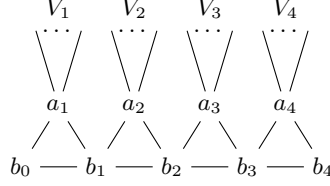


Figure 1: Constraint graph  $G$ .

$b_{i+1}, \dots, b_t$  are all set to 1; consequently  $a_i$  is set to 0 and all  $a_j$  for  $j \neq i$  are set to 1. The constraints in group (1) provide that if we set  $a_i$  to 0, then we obtain from  $C'$  the original constraint  $C$ ; if we set  $a_i$  to 1 then we obtain a constraint that can be satisfied by setting all remaining variables to 1. We conclude that  $\{0, 1\}$ -CSP(width) is compositional.

In order to apply Theorem 1, it remains to establish that the unparameterized version of  $\{0, 1\}$ -CSP(width) is NP-complete. Deciding whether a constraint network  $I$  over the universe  $\{0, 1\}$  is satisfiable is well-known to be NP-complete (say by reducing 3-SAT). To a constraint network  $I$  on  $n$  variables we can always add a trivial width  $w = n - 1$  tree decomposition of its constraint graph (taking a single tree node  $t$  where  $\lambda(t)$  contains all variables of  $I$ ). Hence  $\text{UP}[\{0, 1\}\text{-CSP}(\text{width})]$  is NP-complete.  $\square$

## 5 Satisfiability

The *propositional satisfiability problem* (SAT) was the first problem shown to be NP-hard (Cook, 1971). Despite its hardness, SAT solvers are increasingly leaving their mark as a general-purpose tool in areas as diverse as software and hardware verification, automatic test pattern generation, planning, scheduling, and even challenging problems from algebra (Gomes et al., 2008). SAT solvers are capable of exploiting the hidden structure present in real-world problem instances. The concept of *backdoors*, introduced by Williams, Gomes, and Selman (2003) provides a means for making the vague notion of a hidden structure explicit. Backdoors are defined with respect to a “sub-solver” which is a polynomial-time algorithm that correctly decides the satisfiability for a class  $\mathcal{C}$  of CNF formulas. More specifically, Gomes et al. (2008) define a *sub-solver* to be an algorithm  $\mathcal{A}$  that takes as input a CNF formula  $F$  and has the following properties: (i) *Trichotomy*:  $\mathcal{A}$  either rejects the input  $F$ , or determines  $F$  correctly as unsatisfiable or satisfiable; (ii) *Efficiency*:  $\mathcal{A}$  runs in polynomial time; (iii) *Trivial Solvability*:  $\mathcal{A}$  can determine if  $F$  is trivially satisfiable (has no clauses) or trivially unsatisfiable (contains only the empty clause); (iv.) *Self-Reducibility*: if  $\mathcal{A}$  determines  $F$ , then for any variable  $x$  and value  $\varepsilon \in \{0, 1\}$ ,  $\mathcal{A}$  determines  $F[x = \varepsilon]$ .  $F[\tau]$  denotes the formula obtained from  $F$  by applying the partial assignment  $\tau$ , i.e., satisfied clauses are removed and false literals are removed from the remaining clauses.

We identify a sub-solver  $\mathcal{A}$  with the class  $\mathcal{C}_{\mathcal{A}}$  of CNF formulas whose satisfiability can be determined by  $\mathcal{A}$ . A *strong  $\mathcal{A}$ -backdoor set* (or  *$\mathcal{A}$ -backdoor*, for short) of a CNF formula  $F$  is a set  $B$  of variables such that for each possible truth assignment  $\tau$  to the variables in  $B$ , the satisfiability of  $F[\tau]$  can be determined by sub-solver  $\mathcal{A}$  in time  $O(n^c)$ . Hence, if we know an  $\mathcal{A}$ -backdoor of size  $k$ , we can decide the satisfiability of  $F$  by running  $\mathcal{A}$  on  $2^k$  instances  $F[\tau]$ , yielding a time bound of  $O(2^k n^c)$ . Hence SAT decision is fixed-parameter tractable in the backdoor size  $k$  for any sub-solver  $\mathcal{A}$ . Hence the following problem is clearly fixed-parameter tractable for any sub-solver  $\mathcal{A}$ .

**SAT( $\mathcal{A}$ -backdoor)**

*Instance*: A CNF formula  $F$ , and an  $\mathcal{A}$ -backdoor  $B$  of  $F$  of size  $k$ .

*Parameter*: The integer  $k$ .

*Question*: Is  $F$  satisfiable?

We are concerned with the question of whether instead of trying all  $2^k$  possible partial assignments we can reduce the instance to a polynomial kernel. We will establish a very general result that applies to all possible sub-solvers.

**Theorem 4.** *SAT( $\mathcal{A}$ -backdoor) does not admit a polynomial kernel for any sub-solver  $\mathcal{A}$  unless the Polynomial Hierarchy collapses.*

*Proof.* We will devise polynomial parameter transformations from the following parameterized problem which is known to be compositional (Fortnow and Santhanam, 2008) and therefore unlikely to admit a polynomial kernel.

**SAT(vars)**

*Instance:* A propositional formula  $F$  in CNF on  $n$  variables.

*Parameter:* The number  $n$  of variables.

*Question:* Is  $F$  satisfiable?

Let  $F$  be a CNF formula and  $V$  the set of all variables of  $F$ . Due to property (ii) of a sub-solver,  $V$  is an  $\mathcal{A}$ -backdoor set for any  $\mathcal{A}$ . Hence, by mapping  $(F, n)$  (as an instance of **SAT(vars)**) to  $(F, V)$  (as an instance of **SAT( $\mathcal{A}$ -backdoor)**) provides a (trivial) polynomial parameter transformation from **SAT(vars)** to **SAT( $\mathcal{A}$ -backdoor)**. Since the unparameterized versions of both problems are clearly NP-complete, the result follows by Theorem 2.  $\square$

Let **3SAT( $\pi$ )** (where  $\pi$  is an arbitrary parameterization) denote the problem **SAT( $\pi$ )** restricted to 3CNF formula, i.e., to CNF formulas where each clause contains at most three literals. In contrast to **SAT(vars)**, the parameterized problem **3SAT(vars)** has a trivial polynomial kernel: if we remove duplicate clauses, then any 3CNF formula on  $n$  variables contains at most  $O(n^3)$  clauses, and so is a polynomial kernel. Hence the easy proof of Theorem 4 does not carry over to **3SAT( $\mathcal{A}$ -backdoor)**. We therefore consider the cases **3SAT(HORN-backdoor)** and **3SAT(2CNF-backdoor)** separately, these cases are important since the detection of HORN and 2CNF-backdoors is fixed-parameter tractable (Nishimura, Ragde, and Szeider, 2004).

**Theorem 5.** *Neither **3SAT(HORN-backdoor)** nor **3SAT(2CNF-backdoor)** admit a polynomial kernel unless the Polynomial Hierarchy collapses.*

*Proof.* Let  $\mathcal{C} \in \{\text{HORN}, 2\text{CNF}\}$ . We show that **3SAT( $\mathcal{C}$ -backdoor)** is compositional. Let  $(F_i, B_i)$ ,  $1 \leq i \leq t$ , be a given sequence of instances of **3SAT( $\mathcal{C}$ -backdoor)** where  $F_i$  is a 3CNF formula and  $B_i$  is a  $\mathcal{C}$ -backdoor set of  $F_i$  of size  $k$ . We distinguish two cases.

Case 1:  $t > 2^k$ . Let  $\|F_i\| := \sum_{C \in F_i} |C|$  and  $n := \max_{i=1}^t \|F_i\|$ . Whether  $F_i$  is satisfiable or not can be decided in time  $O(2^k n)$  since the satisfiability of a Horn or 2CNF formula can be decided in linear time. We can check whether at least one of the formulas  $F_1, \dots, F_t$  is satisfiable in time  $O(t 2^k n) \leq O(t^2 n)$  which is polynomial in  $t + n$ . If some  $F_i$  is satisfiable, we output  $(F_i, B_i)$ ; otherwise we output  $(F_1, B_1)$  ( $F_1$  is unsatisfiable). Hence we have a composition algorithm.

Case 2:  $t \leq 2^k$ . This case is more involved. We construct a new instance  $(F, B)$  of **3SAT( $\mathcal{C}$ -backdoor)** as follows.

Let  $s = \lceil \log_2 t \rceil$ . Since  $t \leq 2^k$ ,  $s \leq k$  follows.

Let  $V_i$  denote the set of variables of  $F_i$ . We may assume, w.l.o.g., that  $B_1 = \dots = B_t$  and that  $V_i \cap V_j = B_1$  for all  $1 \leq i < j \leq t$  since otherwise we can change names of variable accordingly. In a first step we obtain from every  $F_i$  a CNF formula  $F'_i$  as follows. For each variable  $x \in V_i \setminus B_1$  we take two new variables  $x_0$  and  $x_1$ . We replace each positive occurrence of a variable  $x \in V_i \setminus B_1$  in  $F_i$  with the literal  $x_0$  and each negative occurrence of  $x$  with the literal  $\neg x_s$ . We add all clauses of the form  $(\neg x_{j-1} \vee x_j)$  for  $1 \leq j \leq s$ ; we call these clauses “*connection clauses*.” Let  $F'_i$  be the formula obtained from  $F_i$  in this way. We observe that  $F'_i$  and  $F_i$  are SAT-equivalent, since the connection clauses form an implication chain. Since the connection clauses are both Horn and 2CNF,  $B_1$  is also a  $\mathcal{C}$ -backdoor of  $F'_i$ .

We take a set  $Y = \{y_1, \dots, y_s\}$  of new variables. Let  $C_1, \dots, C_{2^s}$  be the sequence of all  $2^s$  possible clauses (modulo permutation of literals within a clause) containing exactly  $s$  literals over the variables in  $Y$ . Consequently we can write  $C_i$  as  $(\ell_1^i \vee \dots \vee \ell_s^i)$  where  $\ell_j^i \in \{y_i, \neg y_i\}$ .

For  $1 \leq i \leq t$  we add to each connection clause  $(\neg x_{j-1} \vee x_j)$  of  $F'_i$  the literal  $\ell_j^i \in C_i$ . Let  $F''_i$  denote the 3CNF formula obtained from  $F'_i$  this way.

For  $t < i \leq 2^s$  we define 3CNF formulas  $F''_i$  as follows. If  $s \leq 3$  then  $F''_i$  consists just of the clause  $C_i$ . If  $s > 3$  then we take new variables  $z_2^i, \dots, z_{s-2}^i$  and let  $F''_i$  consist of the clauses  $(\ell_1^i \vee \ell_2^i \vee \neg z_2^i)$ ,  $(\ell_3^i \vee z_2^i \vee \neg z_3^i)$ ,  $\dots$ ,  $(\ell_{s-2}^i \vee z_{s-3}^i \vee \neg z_{s-2}^i)$ ,  $(\ell_{s-1}^i \vee \ell_s^i \vee z_{s-2}^i)$ . Finally, we let  $F$  be the 3CNF formula containing all the clauses from  $F''_1, \dots, F''_{2^s}$ . Any assignment  $\tau$  to  $Y \cup B_1$  that satisfies  $C_i$  can be extended to an assignment that satisfies  $F''_i$  since such assignment satisfies at least one connection clause  $(x_{j-1} \vee x_j \vee \ell_j^i)$  and so the chain of implications from  $x_o$  to  $x_s$  is broken.

It is not difficult to verify the following two claims. (i)  $F$  is satisfiable if and only if at least one of the formulas  $F_i$  is satisfiable. (ii)  $B = Y \cup B_1$  is a  $\mathcal{C}$ -backdoor of  $F$ . Hence we have also a composition algorithm in Case 2, and thus **3SAT**( $\mathcal{C}$ -backdoor) is compositional. Clearly **UP**[**3SAT**( $\mathcal{C}$ -backdoor)] is NP-complete, hence the result follows from Theorem 1.  $\square$

## 6 Global Constraints

The success of today's constraint solvers relies heavily on efficient algorithms for special purpose *global constraints* (van Hoeve and Katriel, 2006). A global constraint specifies a pattern that frequently occurs in real-world problems, for instance, it is often required that variables must all take different values (e.g., activities requiring the same resource must all be assigned different times). The **ALLDIFFERENT** global constraint efficiently encodes this requirement.

More formally, a global constraint is defined for a set  $S$  of variables, each variable  $x \in S$  ranges over a finite domain  $dom(x)$  of values. An *instantiation* is an assignment  $\alpha$  such that  $\alpha(x) \in dom(x)$  for each  $x \in S$ . A global constraint defines which instantiations are legal and which are not. A global constraint is *consistent* if it has at least one legal instantiation, and it is *domain consistent* (or hyper arc consistent) if for each variable  $x \in S$  and each value  $d \in dom(x)$  there is a legal instantiation  $\alpha$  with  $\alpha(x) = d$ . For all global constraints considered in this paper, domain consistency can be reduced to a quadratic number of consistency checks, hence we will focus on consistency. We assume that the size of a representation of a global constraint is polynomial in  $\sum_{x \in S} |dom(x)|$ .

For several important types  $\mathcal{T}$  of global constraints, the problem of deciding whether a constraint of type  $\mathcal{T}$  is consistent (in symbols  $\mathcal{T}$ -**Cons**) is NP-hard. Examples for such intractable types of constraints are **NVALUE**, **DISJOINT**, and **USES** (Bessière et al., 2004). An **NVALUE** constraint over a set  $X$  of variables requires from a legal instantiation  $\alpha$  that  $|\{\alpha(x) : x \in X\}| = N$ ; **ALLDIFFERENT** is the special case where  $N = |X|$ . The global constraints **DISJOINT** and **USES** are specified by two sets of variables  $X, Y$ ; **DISJOINT** requires that  $\alpha(x) \neq \alpha(y)$  for each pair  $x \in X$  and  $y \in Y$ ; **USES** requires that for each  $x \in X$  there is some  $y \in Y$  such that  $\alpha(x) = \alpha(y)$ . For a set  $X$  of variables we write  $dom(X) = \bigcup_{x \in X} dom(x)$ .

Bessière et al. (2008) considered  $dx = |dom(X)|$  as parameter for **NVALUE**,  $dxy = |dom(X) \cap dom(Y)|$  as parameter for **DISJOINT**, and  $dy = |dom(Y)|$  as parameter for **USES**. They showed that consistency checking is fixed-parameter tractable for the constraints under the respective parameterizations, i.e., the problems **NVALUE-CONS**( $dx$ ), **DISJOINT-CONS**( $dxy$ ), and **USES-CONS**( $dy$ ) are fixed-parameter tractable. We show that it is unlikely that their results can be improved in terms of polynomial kernels.

**Theorem 6.** *The problems **NVALUE-CONS**( $dx$ ), **DISJOINT-CONS**( $dxy$ ), **USES-CONS**( $dy$ ) do not admit polynomial kernels unless the Polynomial Hierarchy collapses.*

*Proof.* We devise a polynomial parameter reduction from **SAT**(vars). We use a construction of Bessière et al. (2004). Let  $F = \{C_1, \dots, C_m\}$  be a CNF formula over variables  $x_1, \dots, x_n$ . We consider the clauses and variables of  $F$  as the variables of a global constraint with domains  $dom(x_i) = \{-i, i\}$ , and  $dom(C_j) = \{i : x_i \in C_j\} \cup \{-i : \neg x_i \in C_j\}$ . Now  $F$  can be encoded as an **NVALUE** constraint with  $X = \{x_1, \dots, x_n, C_1, \dots, C_m\}$  and  $N = n$  (clearly  $F$  is satisfiable if and only if the constraint is consistent). Since  $dx = 2n$  we have a polynomial parameter reduction from **SAT**(vars) to **NVALUE-CONS**( $dx$ ). Similarly, as observed by Bessière et al. (2009),  $F$  can be encoded as a **DISJOINT** constraint with  $X = \{x_1, \dots, x_n\}$  and  $Y = \{C_1, \dots, C_m\}$  ( $dxy \leq 2n$ ), or as a **USES** constraint with  $X = \{C_1, \dots, C_m\}$  and  $Y = \{x_1, \dots, x_n\}$  ( $dy = 2n$ ). Since the unparameterized problems are clearly NP-complete, the result follows by Theorem 2.  $\square$

Further results on kernels for global constraints have been obtained by Gaspers and Szeider (2011).

## 7 Bayesian Reasoning

*Bayesian networks* (BNs) have emerged as a general representation scheme for uncertain knowledge (Pearl, 2010). A BN models a set of stochastic variables, the independencies among these variables, and a joint probability distribution over these variables. For simplicity we consider the important special case where the stochastic variables are Boolean. The variables and independencies are modelled in the BN by a directed acyclic graph  $G = (V, A)$ , the joint probability distribution is given by a table  $T_v$  for each node  $v \in V$  which defines a probability  $T_{v|U}$  for each possible instantiation  $U = (d_1, \dots, d_s) \in \{\text{true}, \text{false}\}^s$  of the parents  $v_1, \dots, v_s$  of  $v$  in  $G$ . The probability  $\Pr(U)$  of a complete instantiation  $U$  of the variables of  $G$  is given by the product of  $T_{v|U}$  over all variables  $v$ . We consider the problem **POSITIVE-BN-INFERENCE** which takes as input a Boolean BN  $(G, T)$  and a variable  $v$ , and asks whether  $\Pr(v = \text{true}) > 0$ . The problem is NP-complete (Cooper, 1990) and moves from NP to #P if we ask to compute  $\Pr(v = \text{true})$  (Roth, 1996). The problem can be solved in polynomial time if the BN is *singly connected*, i.e, if there is at most one undirected path between any two variables (Pearl, 1988). It is natural to parametrize the problem by the number of variables one must delete in order to make the BN singly connected (the deleted variables form a *loop cutset*). In fact, **POSITIVE-BN-INFERENCE**(loop cutset size) is easily seen to be fixed-parameter tractable as we can determine whether  $\Pr(v = \text{true}) > 0$  by taking the maximum of  $\Pr(v = \text{true} \mid U)$  over all  $2^k$  possible instantiations of the  $k$  cutset variables, each of which requires processing of a singly connected network. However, although fixed-parameter tractable, it is unlikely that the problem admits a polynomial kernel.

**Theorem 7.** **POSITIVE-BN-INFERENCE**(loop cutset size) *does not admit a polynomial kernel unless the Polynomial Hierarchy collapses.*

*Proof.* (Sketch.) We give a polynomial parameter transformation from **SAT**(vars) and apply Theorem 2. The reduction is based on the reduction from 3SAT given by Cooper (1990). However, we need to allow clauses with an arbitrary number of literals since, as observed above, **3SAT**(vars) has a polynomial kernel. Let  $F$  be a CNF formula on  $n$  variables. We construct a BN  $(G, T)$  such that for a variable  $v$  we have  $\Pr(v = \text{true}) > 0$  if and only if  $F$  is satisfiable. Cooper uses *input nodes*  $u_i$  for representing variables of  $F$ , *clause nodes*  $c_i$  for representing the clauses of  $F$ , and *conjunction nodes*  $d_i$  for representing the conjunction of the clauses. We proceed similarly, however, we cannot represent a clause of large size with a single clause node  $c_i$ , as the required table  $T_{c_i}$  would be of exponential size. Therefore we split clauses containing more than 3 literals into several clause nodes, as indicated in Figure 2. It remains to observe that

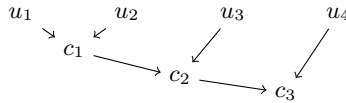


Figure 2: BN representation of a clause on four literals.

the set of input nodes  $E = \{u_1, \dots, u_n\}$  is a loop cutset of the constructed BN, hence we have indeed a polynomial parameter transformation from **SAT**(vars) to **POSITIVE-BN-INFERENCE**(loop cutset size). The result follows by Theorem 2.  $\square$

## 8 Nonmonotonic Reasoning

*Logic programming with negation* under the stable model semantics is a well-studied form of nonmonotonic reasoning (Gelfond and Lifschitz, 1988; Marek and Truszczyński, 1999). A (normal) *logic program*  $P$  is a finite set of rules  $r$  of the form

$$h \leftarrow a_1 \wedge \dots \wedge a_m \wedge \neg b_1 \wedge \dots \wedge \neg b_n$$

where  $h, a_i, b_i$  are *atoms*, where  $h$  forms the head and the  $a_i, b_i$  from the body of  $r$ . We write  $H(r) = h$ ,  $B^+(r) = \{a_1, \dots, a_m\}$ , and  $B^-(r) = \{b_1, \dots, b_n\}$ . Let  $I$  be a finite set of atoms. The *GF reduct*  $P^I$  of



a logic program  $P$  under  $I$  is the program obtained from  $P$  by removing all rules  $r$  with  $B^-(r) \cap I \neq \emptyset$ , and removing from the body of each remaining rule  $r'$  all literals  $\neg b$  with  $b \in I$ .  $I$  is a *stable model* of  $P$  if  $I$  is a minimal model of  $P^I$ , i.e., if (i) for each rule  $r \in P^I$  with  $B^+(r) \subseteq I$  we have  $H(r) \in I$ , and (ii) there is no proper subset of  $I$  with this property. The *undirected dependency graph*  $U(P)$  of  $P$  is formed as follows. We take the atoms of  $P$  as vertices and add an edge  $x - y$  between two atoms  $x, y$  if there is a rule  $r \in P$  with  $H(r) = x$  and  $y \in B^+(r)$ , and we add a path  $x - u - y$  if  $H(r) = x$  and  $y \in B^-(r)$  ( $u$  is a new vertex of degree 2). The *feedback width* of  $P$  is the size of a smallest set  $V$  of atoms such that every cycle of  $U(P)$  runs through an atom in  $V$ .

A fundamental computational problem is **Stable Model Existence (SME)**, which asks whether a given normal logic program has a stable model. The problem is well-known to be NP-complete (Marek and Truszczyński, 1991). Gottlob, Scarcello, and Sideri (2002) showed that **SME(feedback width)** is fixed-parameter tractable (see Fichte and Szeider (2011) for generalizations). We show that this result cannot be strengthened with respect to a polynomial kernel.

**Theorem 8.** *SME(feedback width) does not admit a polynomial kernel unless the Polynomial Hierarchy collapses.*

*Proof.* (Sketch.) We give a polynomial parameter transformation from **SAT(vars)** to **SME(feedback width)** using a construction of Niemelä (1999). Given a CNF formula  $F$  on  $n$  variables, we construct a logic program  $P$  as follows. For each variable  $x$  of  $F$  we take two atoms  $x$  and  $\hat{x}$  and include the rules  $(\hat{x} \leftarrow \neg x)$  and  $(x \leftarrow \neg \hat{x})$ ; for each clause  $C$  of  $F$  we take an atom  $c$  and include for each positive literal  $a$  of  $C$  the rule  $(c \leftarrow a)$ , and for each negative literal  $\neg a$  of  $C$  the rule  $(c \leftarrow \hat{a})$ ; finally, we take two atoms  $s$  and  $f$  and include the rule  $(f \leftarrow \neg f \wedge \neg s)$  and for each clause  $C$  of  $F$  the rule  $(s \leftarrow c)$ .  $F$  is satisfiable if and only if  $P$  has a stable model (Niemelä, 1999). It remains to observe that each cycle of  $U(P)$  runs through a vertex in  $V = \{x, \hat{x} : x \in \text{vars}(F)\}$ , hence the feedback width of  $P$  is at most  $2n$ . Hence we have a polynomial parameter transformation from **SAT(vars)** to **SME(feedback width)**. The result follows by Theorem 2.  $\square$

## 9 Conclusion

We have established super-polynomial kernel lower bounds for a wide range of important AI problems, providing firm limitations for the power of polynomial-time preprocessing for these problems. We conclude from these results that in contrast to many optimization problems (see Section 1), typical AI problems do not admit polynomial kernels. Our results suggest the consideration of alternative approaches. For example, it might still be possible that some of the considered problems admit polynomially sized Turing kernels, i.e., a polynomial-time preprocessing to a Boolean combination of a polynomial number of polynomial kernels. In the area of optimization, parameterized problems are known that do not admit polynomial kernels but admit polynomial Turing kernels (Fernau et al., 2009). This suggests a theoretical and empirical study of Turing kernels for the AI problems considered.

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